Sorted L-One Penalized Estimation (SLOPE)

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Multiple Testing

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Model selection in multiple regression

- LASSO
- SLOPE
- FDR control under orthogonal design
- Asymptotic minimaxity
- Extensions and applications
- Adaptive Bayesian Slope

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$$\alpha = 0.05, p_0 = 5000 \rightarrow E(V) = 250$$

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Benjamini-Hochberg procedure:

(1) $|X|_{(1)} \geq |X|_{(2)} \geq \ldots \geq |X|_{(p)}$

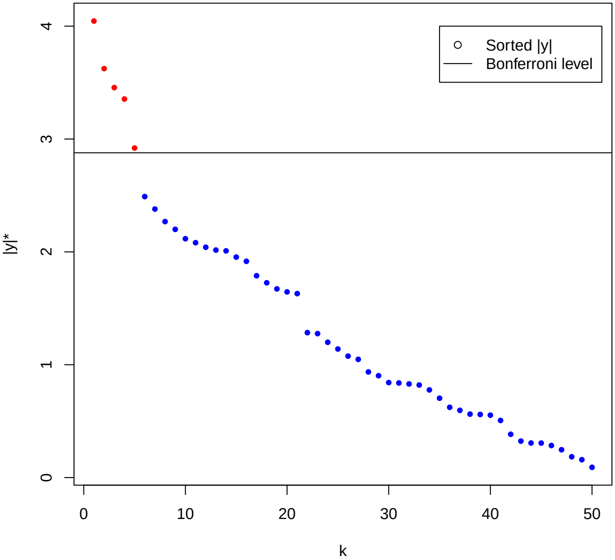
(2) Find the largest index i such that

$$|X|_{(i)} \geq \sigma\Phi^{-1}(1 - \alpha_i), \quad \alpha_i = \alpha\frac{i}{2p}, \tag{1}$$

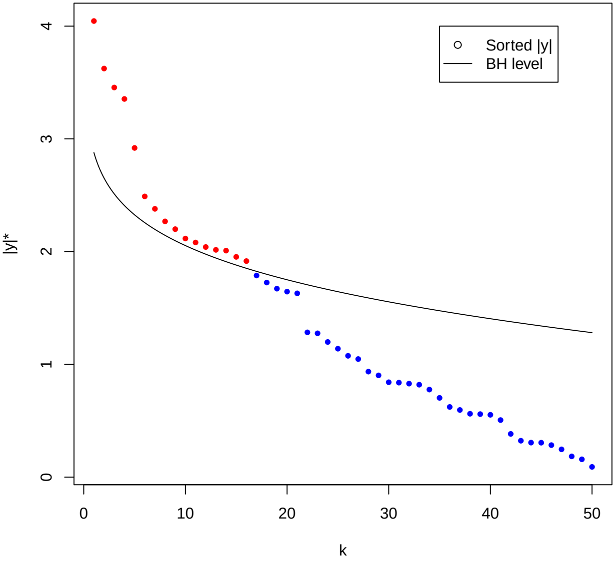
Call this index i_{SU}.

(3) Reject all $H_{(i)}$'s for which $i \leq i_{\mathrm{SU}}$

Bonferroni correction



Benjamini and Hochberg correction



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(Benjamini, Yekutieli, 2001) BH controls FDR at the level $\alpha\frac{p_0}{p}$ if the test statistics are "positively" correlated (Positive Regression Dependency on each from a subset). BH always controls FDR at the level $\alpha\frac{p_0}{p}$ if $|X|_{(i)}$ is compared with $\sigma\Phi^{-1}\left(1 - \frac{i\alpha}{p\sum_{i=1}^{p}\frac{1}{i}}\right)$.

LASSO

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LASSO (BPDN) : Convex program: Minimize $||b||_1$ subject to $||Y - Xb||_2^2 \leq \epsilon$

Or alternatively: $min_{b\in R^p} \frac{1}{2}||y - Xb||_2^2 + \lambda||b||_1$

Selection of the tuning parameter for LASSO

- General rule: the reduction of λ_L results in identification of more elements from the true support (true discoveries) but at the same time it produces more falsely identified variables (false discoveries)
- The choice of λ_L is challenging- e.g. crossvalidation typically leads to many false discoveries
- When $X^T X = I$ Lasso selects X_j iff $|\hat{\beta}_j^{LS}| > \lambda$
- Selection $\lambda = \sigma\Phi^{-1}(1 - \alpha/(2p)) \approx \sigma\sqrt{2\log p}$ corresponds to Bonferroni correction and controls FWER.

Sorted L-One Penalized Estimation

M. Bogdan, E. van den Berg, C. Sabatti, W. Su, E. J. Candès, AOAS 2015

SLOPE (2)

SLOPE estimate is given by

$$\min_{b \in \mathbb{R}^p} \quad \frac{1}{2}\|y - Xb\|_{\ell_2}^2 + \lambda_1 |b|_{(1)} + \lambda_2 |b|_{(2)} + \cdots + \lambda_p |b|_{(p)},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$ and $|b|_{(1)} \geq |b|_{(2)} \geq \cdots \geq |b|_{(p)}$

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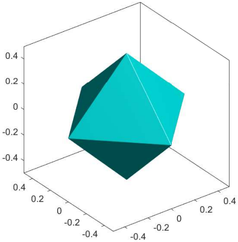
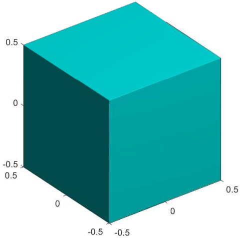
R package *SLOPE* available on CRAN, by E. Patterson

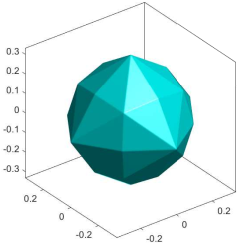
Unit balls for different SLOPE sequences



(a) (2,2,2)



(b) (2,0,0)



(c) (3,2,1)

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- the number of false discoveries, $V := \left|\left\{i : \ \beta_i = 0, \quad \widetilde{\beta}_i \neq 0\right\}\right|$
- false discovery rate, $FDR := \mathbb{E}\left[\frac{V}{\max\{R,1\}}\right]$

FDR control with SLOPE

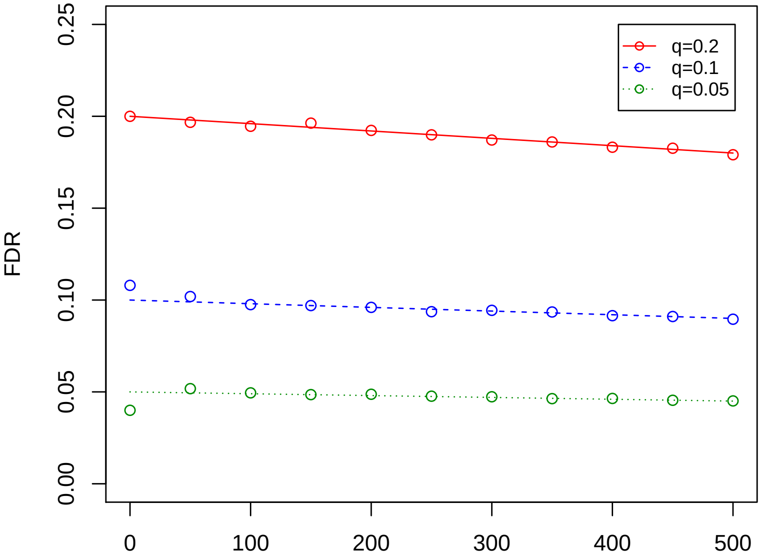
Theorem (B,van den Berg, Sabatti, Su and Candès (2015))

When $X^TX = I$ SLOPE with

$$\lambda_i := \sigma\Phi^{-1}\left(1 - i \cdot \frac{q}{2p}\right)$$

controls FDR at the level $q\frac{p_0}{p}$.

Orthogonal design, $n = p = 5000$



Asymptotic optimality, Su and Candès (Annals of Statistics, 2016)

Theorem

Let $X_{ij} \sim N(0, 1/\sqrt{n})$. *Fix* $0 < q < 1$ *and choose* $\lambda = \sigma(1+\epsilon)\lambda^{BH}(q)$ *for some arbitrary constant* $0 < \epsilon < 1$. *Suppose* $k/p \to 0$ *and* $\frac{k \log p}{n} \to 0$. *Then*

$$\sup_{||\beta_0|| \leq k} P\left(\frac{||\hat{\beta}_{SL} - \beta||^2}{2\sigma^2 k \log(p/k)} > 1 + 3\epsilon\right) \to 0$$

$$\inf_{\hat{\beta}} sup_{||\beta_0|| \leq k} P\left(\frac{||\hat{\beta} - \beta||^2}{2\sigma^2 k \log(p/k)} > 1 - \epsilon\right) \to 1$$

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Minimax estimation/prediction rate $[k \log(p/k)]$ under weighted restricted eigenvalue condition (large collection of random matrices)

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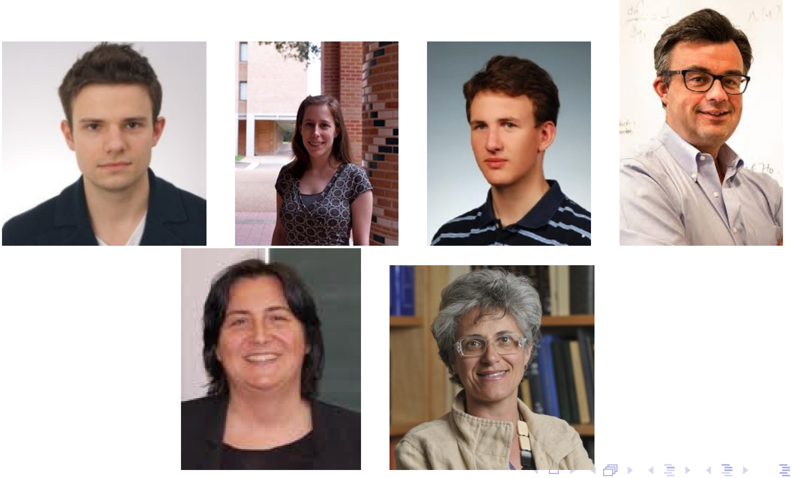
LASSO rate of convergence - $k\log(p)$

Genome Wide Association Studies

D. Brzyski, C.B. Peterson, P.Sobczyk, E.J. Candès, M. Bogdan, C. Sabatti, (Genetics, 2017)
R package *geneSLOPE* available on CRAN, by P. Sobczyk



Computer simulations

Genotypes - the North Finland Birth Cohort (NFBC) dataset.

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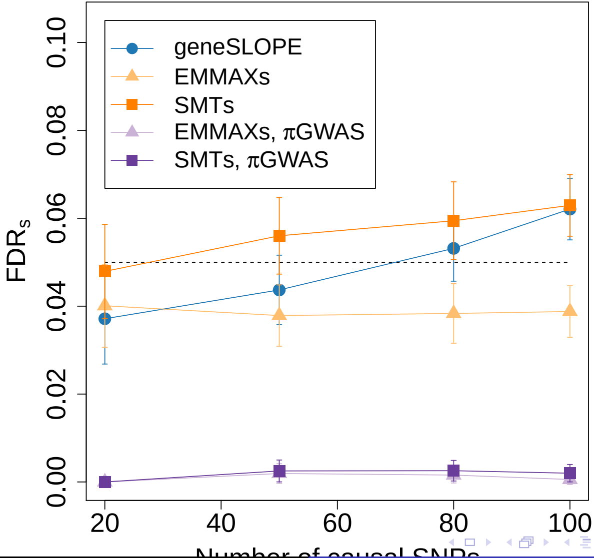
Support randomly selected for every replication from the set of well separated SNPs

FDR


geneSLOPE
EMMAXs
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0.00
0.02
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Number of causal SNPs

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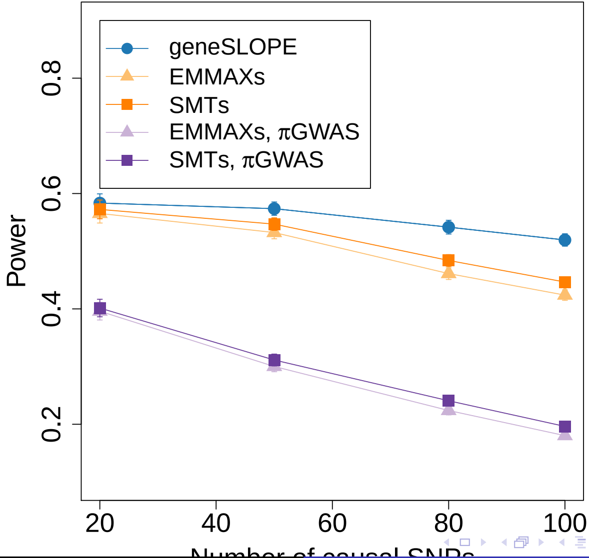
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Group SLOPE, (Brzyski, Gossmann, Su and Bogdan, JASA, 2019)

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group - additive and dominance effect of a given SNP

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Scenario 1: $Y = X\beta + z$ - additive model

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$$Z_{ij} = \begin{cases} -1 & \text{for} \quad aa, AA \\ 1 & \text{for} \quad aA \end{cases}, \tag{4}$$

$$y = [X, Z][\beta'_X, \beta'_Z]' + \epsilon \quad .$$

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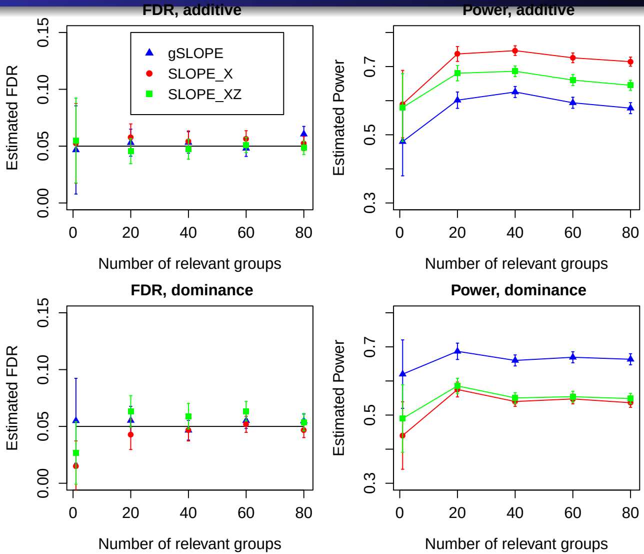
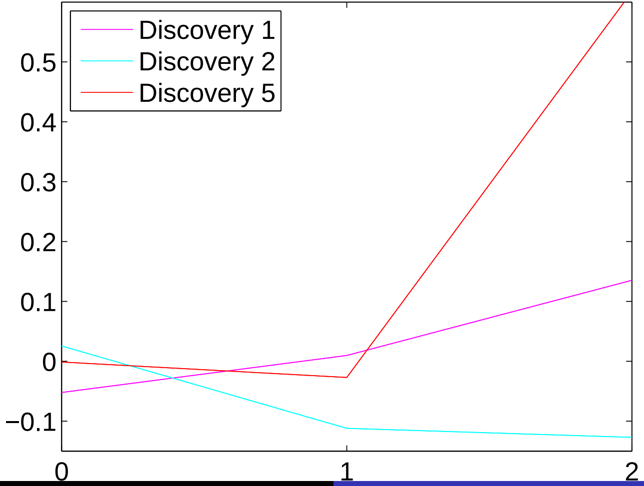
Simulation results



Genes Influencing Level of Triglicerides

5 new discoveries with group SLOPE - recessive rare genetic variants. Discovery 5 - 37 rare homozygotes



Robust regression with SLOPE

A.Virouleau, A.Guilloux, S.Gaiffas, M.Bogdan (arxive, 2017)

Robust regression (2)

Mean shift model: Candes and Randall (2006), Gannaz (2006) and McCann and Welsch (CSDA, 2007) ,

$$y = X\beta + I\mu + \varepsilon \tag{6}$$

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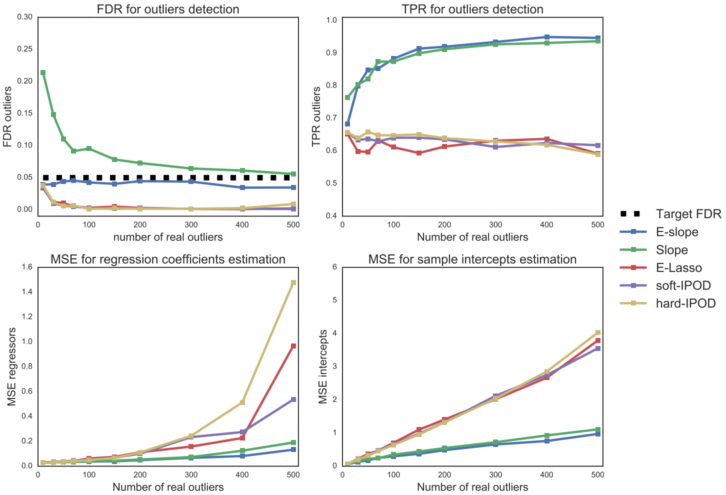
$$\min_{\beta \in \mathbb{R}^p, \mu \in \mathbb{R}^n} \left\{ \|y - X\beta - \mu\|_2^2 + 2\rho_1 J_{\tilde{\lambda}}(\beta) + 2\rho_2 J_\lambda(\mu) \right\}$$

Low dimensional set-up; small outliers



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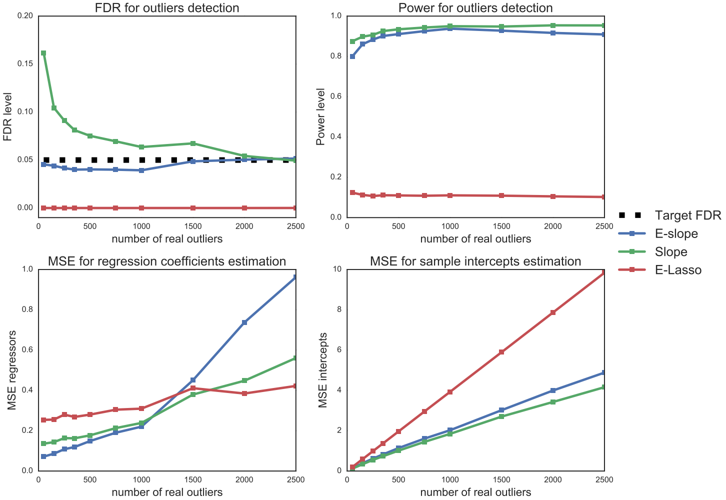
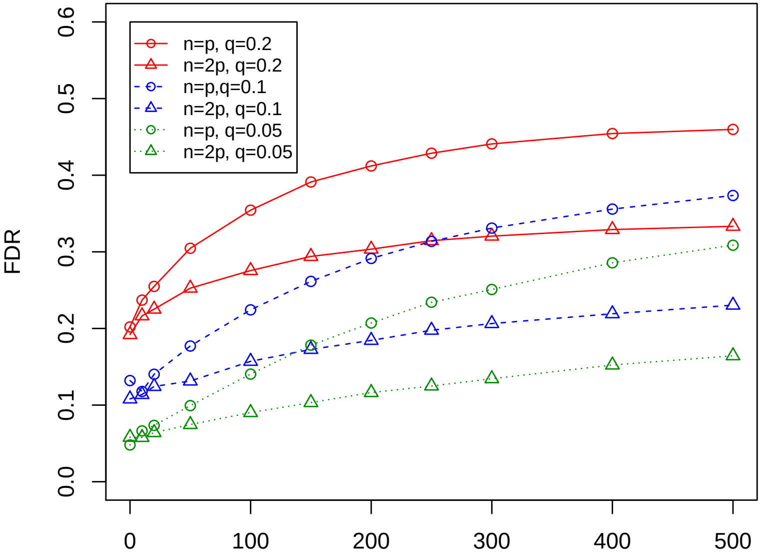


Adaptive SLOPE

W. Jiang, B. Miasojedow, V. Rockova, J. Josse, M. Bogdan (in progress)

Gaussian design (1), $n = p = 5000$



Problems with FDR control

Similar problems occur for LASSO: Precise FDR-Power Tradeoff under asymptotic asymptotic assumptions of AMP theory provided in (Su, B, Candès, AOS 2017)

Intuitive explanation:

$$\hat{\beta} = \eta_\lambda(\beta_i + X_i' z + v_i)$$

$$v_i = \langle X_i, \sum_{j \neq i} X_j(\beta_j - \hat{\beta}_j) \rangle$$

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When the design is not orthogonal: $v_i \neq 0$ - additional noise, dependent on λ (level of shrinkage), the level of sparsity and magnitude of true signals

Adaptive LASSO and Bayesian Interpretation of LASSO

Adaptive LASSO [Zou, *JASA* 2006], [Candès, Wakin and Boyd, *J. Fourier Anal. Appl.* 2008]

$$\beta_{aL} = argmin_b \left\{ \frac{1}{2} \|y - Xb\|_2^2 + \lambda \sum_{i=1}^{p} w_i |b|_i \right\}, \qquad (7)$$

where $w_i = \frac{1}{\hat{\beta}_i}$, and $\hat{\beta}_i$ is some consistent estimator of β_i.

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where $w_i = \frac{1}{\hat{\beta}_i}$, and $\hat{\beta}_i$ is some consistent estimator of β_i.

LASSO has a Bayesian interpretation as a posterior mode under the prior

$$\pi(\beta) = C(\lambda) \prod_{i=1}^{n} \mathrm{e}^{-|\beta_i|\lambda}$$

Spike and Slab LASSO (Rockova, George, 2018)

Model for SSL: $\gamma_i = 1$ if i^{th} variable is a signal
Prior for β is given by

$$\pi(\beta|\lambda,\gamma) \propto c^{\sum_{i=1}^{p} 1(\gamma_i=1)} \prod_{i=1}^{p} \mathrm{e}^{-w_i|\beta_i|\lambda_0},$$

where $w_i = 1$ if $\gamma_i = 0$ and $w_i = c \in (0,1)$ if $\gamma_i = 1$.

Spike and Slab LASSO (2)

The maximum aposteriori rule is given by reweighted LASSO

$$\hat{\beta}(\gamma) = argmin_{b \in R^p} \frac{1}{2} ||y - Xb||_2^2 + \lambda \sum_{i=1}^{p} w_i |b_i|$$

$$w_i = c\gamma_i + (1 - \gamma_i)$$

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Prior for γ: $\gamma_1, \ldots, \gamma_p$ are iid such that

$$P(\gamma_i = 1) = \theta = 1 - P(\gamma_i = 0)$$

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In consecutive iterations of SSL γ_i is replaced with

$$\pi_i^t = P(\gamma_i = 1|\beta^t, c) = \frac{c\theta e^{-c|\beta_i^t|\lambda_0}}{c\theta e^{-c|\beta_i^t|\lambda_0} + (1-\theta)e^{-|\beta_i^t|\lambda_0}}$$

and then a new estimate $\hat{\beta}^{t+1}$ is calculated by solving reweighted LASSO with the vector γ replaced with the vector π^t.

Adaptive SLOPE with missing values

Prior for β is given by

$$\pi(\beta|\gamma, c, \sigma^2) \propto c^{\sum_{i=1}^{n} 1(\gamma_i=1)} \prod_{i=1}^{n} \mathrm{e}^{-w_i|\beta_i|\lambda_{r(W\beta,i)}},$$

where W is the diagonal matrix with $W_{ii} = w_i$ and $\lambda = \lambda^{BH}$

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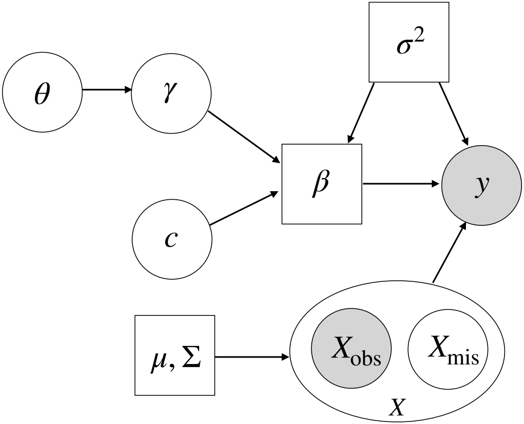
where W is the diagonal matrix with $W_{ii} = w_i$ and $\lambda = \lambda^{BH}$

Missing at Random (MAR) mechanism under assumption $X_i = (X_{i1}, \ldots, X_{ip})$ is normally distributed:

$$X_i \ \text{iid} \ \mathcal{N}_p(\mu, \Sigma), \quad i = 1, \cdots, n.$$

Graphical model of ABSLOPE



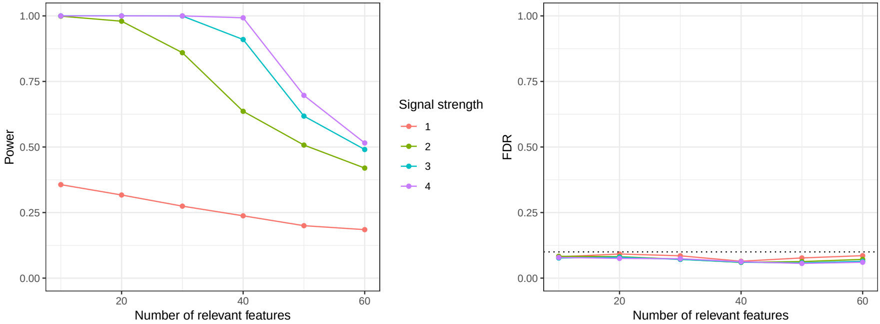
Stochastic approximation EM algorithm

- $\pi(\theta)$ - B(a,b), $\pi(c)$ - $U(0,1)$
- Gibbs sampling of latent variables : $\theta, c, \gamma, c, X_{mis}$
- Estimate parameters $\beta, \sigma, \mu, \Sigma$ by maximizing the complete-data likelihood with sampled values for the latent variables
- When $p > n$, Σ is estimated using the shrinkage estimator of Ledoit and Wolf (2004)
- Approximation of SAEM: ψ,

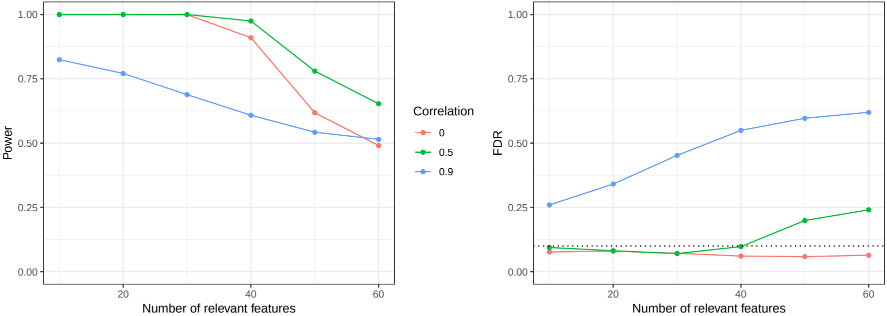
$$\psi^{t+1} = \psi^t + \eta_t \left[\hat{\psi}^t_{MLE} - \psi^t\right],$$

$\eta^t = 1$ for $t \in \{1, \dots, t_0\}$ and $\eta^t = \frac{1}{t-t_0}$ for $t > t_0$

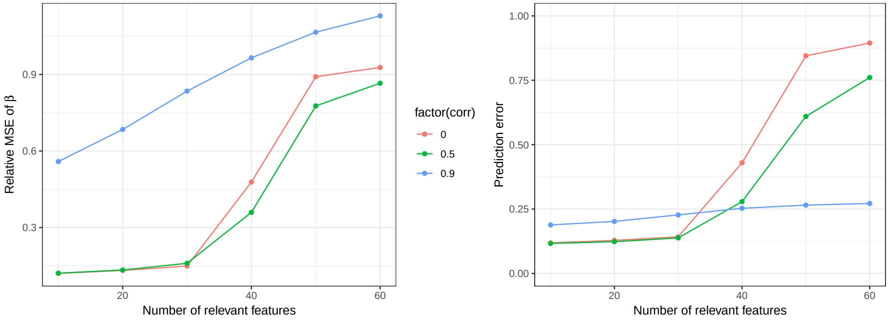
$n = p = 500$, $\rho = 0$, $Na = 10\%$, Power and FDR



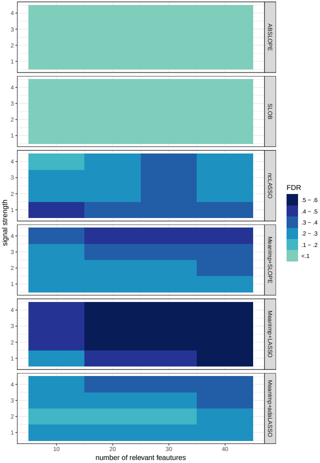
$n = p = 500$, $NA = 10\%$, strong signal, Power and FDR



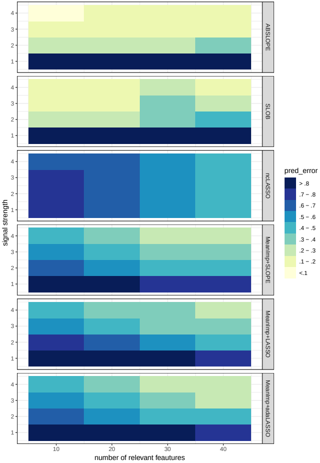
$n = p = 500$, $NA = 10\%$, strong signal, $MSE(\hat{\beta})$, $MSE(X\hat{\beta})$



500×500 dataset, 10% missingness, $\rho = 0.5$



(j) FDR



(k) Prediction error

Variables in the TraumaBase data set (APHP)

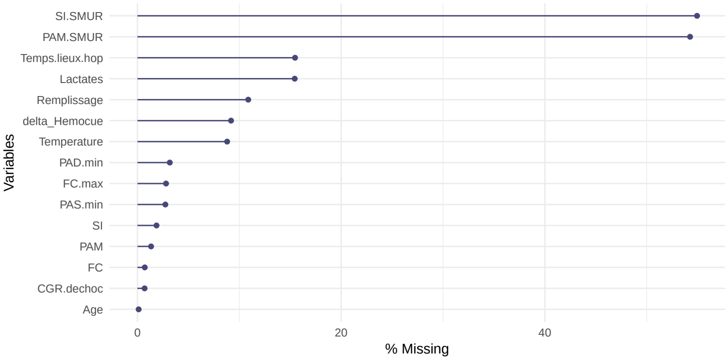
Goal - quick prediction of the level of platelets

- *Age:* Age
- *SI:* Shock index indicates level of occult shock based on heart rate (FC) and systolic blood pressure (PAS). $SI = \frac{FC}{PAS}$. Evaluated on arrival of hospital.
- *PAM:* Mean arterial pressure is an average blood pressure in an individual during a single cardiac cycle, based on systolic blood pressure (PAS) and diastolic blood pressure (PAD). $PAM = \frac{2PAD+PAS}{3}$. Evaluated on arrival of hospital.
- *delta_Hemocue:* The difference between the hemoglobin on arrival at hospital and that in the ambulance.
- *Temps.lieux.hop:* Time spent in hospital *i.e.*, medicalization time, in minutes.
- *Lactates:* The conjugate base of lactic acid.
- *Temperature:* Patient's body temperature.

Variables

- *FC:* heart rate measured on arrival of hospital.
- *Remplissage:* A volume expander is a type of intravenous therapy that has the function of providing volume for the circulatory system.
- *CGR.dechoc:* A binary index which indicates whether the transfusion of Red Blood Cells Concentrates is performed.
- *SI.SMUR:* Shock index measured on ambulance.
- *PAM.SMUR:* Mean arterial pressure measured in the ambulance.
- *FC.max:* Maximum value of measured heart rate in the ambulance.
- *PAS.min:* Minimum value of measured systolic blood pressure in the ambulance.
- *PAD.min:* Minimum value of measured diastolic blood pressure in the ambulance.

Percentage of missing values



Rysunek – Percentage of missing values in each pre-selected variable from TraumaBase.

Selected variables

Method	Variables selected
ABSLOPE	Age $*$ PAM.SMUR, delta_Hemocue $*$ Lactates, Lactates $*$ CGR.dechoc, FC $*$ PAS.min
LASSO	CGR.dechoc, PAS.min, Age $*$ Lactates, Age $*$ Remplissage, delta_Hemocue $*$ Lactates, delta_Hemocue $*$ Remplissage, Lactates $*$ Remplissage, Lactates $*$ CGR.dechoc
adaLASSO	Age $*$ Temps.lieux.hop, Age $*$ FC, Age $*$ PAM.SMUR, Age $*$ PAS.min, PAM $*$ FC, delta_Hemocue $*$ Remplissage, Lactates $*$ Remplissage, FC $*$ FC.max, FC $*$ PAS.min, Remplissage $*$ CGR.dechoc

Prediction error